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Green functions of 2-dimensional Yang-Mills theories on nonorientable surfaces

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Abstract

By using the path integral method , we calculate the Green functions of field strength of Yang-Mills theories on arbitrary nonorientable surfaces in Schwinger-Fock gauge. We show that the non-gauge invariant correlators consist of a free part and an almost x-independent part. We also show that the gauge invariant n-point functions are those corresponding to the free part , as in the case of orientable surfaces.

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It has been long known that the two-dimensional Yang-Mills theory is exactly soluble and indeed locally trivial [1]. The reason for this , at a fundamental level , is that in the two dimensions the Yang-Mills action depends only on the measure μ determined by the metric g. In the recent years these theories has been studied more carefully again.

In refs.[2] and [3] the partition function and the expectation value of Wilson loops has been calculated by means of lattice gauge theory for arbitrary two-dimensional closed Riemann surfaces. These quantities also have been derived in the context of path integral in [4] and [5] and the abelianization technique were used to study this theory in refs.[6] and [7]. There are also some efforts to calculate the explicit expression of the partition function on sphere in small area limit [8,9] and the Wilson loops for $SU(N)$ gauge group [10].

The other interesting quantities that must be calculated are the Green functions of field strength. In ref.[11] some of the correlators have been calculated by the abelianization method and in ref.[12] all n-point functions have been derived by path integral method for arbitrary closed orientable Riemann surfaces. There we have shown that the gauge invariant Green functions correspond to a free field theory.

In this paper we are going to complete our investigation about the correlators of $2d$ Yang-Mills theories by calculating them on arbitrary closed nonorientable surfaces. Such surfaces are connected sums of an orientable suface of genus g with s copies of Klein bottle and r copies of the projective plane ; $\Sigma_{g,s,r}$. $\Sigma_{g,s,r}$ can also be regarded as the connected sum of $r+2(s+g)$ projective planes , provided r and s are not both zero. The procedure that we follow are along that of ref.[12].

To begin , let us first rederive the partition function of Yang-Mills theories on $\Sigma_{g,s,r}$ by path integral method. This quantity has been derived in ref.[3] in the context of lattice gauge theory by using the Migdal's suggestion about the local factor of plaquettes. First we consider $r=0$ case. Consider a genus-g Riemann surfaces with $n=2s$ boundaries. The boundary condition of each boundary loop γ_i is specified by a group element g_i , such that Pexp $\oint_{\gamma_i} A = g_i \in G$, where A is the gauge field and G is an arbitrary non-abelian compact semisimple gauge group. We take the specific case of boundary condition in which the g_i's are $g_1, g_1^{-1}, ..., g_s, g_s^{-1}$. The wave function corresponding to this situation is [4] :

$$\psi(\Sigma_{g,n=2s}, g_1, ..., g_s^{-1}) = \sum_\lambda d(\lambda)^{2-2g-2s}\chi_\lambda(g_1)...\chi_\lambda(g_s^{-1})e^{-\frac{\epsilon}{2}c_2(\lambda)A}. \tag{1}$$

In this relation λ labels the irreducible unitary representation of G , χ_λ is the character , $d(\lambda)$ the dimension and $c_2(\lambda)$ the quadratic Casimir of the representation. A is the area of $\Sigma_{g,n}$ and ϵ is the coupling constant. Note that the Schwinger-Fock gauge was used in calculation of the above wave function [4,12] :

$$A^a_\mu(x) = \int_0^1 dss x^\nu F^a_{\nu\mu}(sx). \tag{2}$$

In the remaining of this paper we will also work in this gauge.

Now if we integrate the eq.(1) over $g_1, ..., g_s$ we will find the partition function of genus $g + s$ orientable surface. But as was mentioned in [3] the action of an orientation-reversing diffeomorphism on boundaries is :

$$\chi_\lambda(g) \longrightarrow \chi_\lambda(g^{-1}) = \chi_{\bar{\lambda}}(g), \tag{3}$$

where $\bar{\lambda}$ is the complex conjugate representation of λ. So if we change $\chi_\lambda(g_i^{-1}) \longrightarrow \chi_{\bar{\lambda}}(g_i^{-1})$ in eq.(1) and then integrate over g_i's , we will recover the partition function of a genus-g surface with s copies of Klein bottle :

$$Z_{g,s,r=0} = \sum_{\lambda=\bar{\lambda}} d(\lambda)^{2-2g-2s} e^{-\frac{\epsilon}{2}c_2(\lambda)A}, \tag{4}$$

which is the same as what obtained in [3] . In the above we have used the following orthogonality relation of characters :

$$\int \chi_\lambda(g)\chi_\mu(g^{-1})dg = \delta_{\lambda\mu}, \tag{5}$$

Now let us consider RP^2. Consider a disk with boundary condition

$$\text{Pexp} \oint_{\partial D} A = g = g'^2. \tag{6}$$

The wave function of this disk is

$$\psi_D(g'^2) = \sum_\lambda d(\lambda)\chi_\lambda(g'^2)e^{-\frac{\epsilon}{2}c_2(\lambda)A}. \tag{7}$$

To obtain the partition function of RP^2 , we just have to integrate over g' , so that we have glued the boundaries in such a manner to obtain RP^2 from the disc. The partition function thus becomes :

$$Z_{RP^2} = \int dg'\psi_D(g'^2) = \sum_\lambda f_\lambda e^{-\frac{\epsilon}{2}c_2(\lambda)A}, \tag{8}$$

where

$$f_\lambda := \int \chi_\lambda(g^2)dg. \tag{9}$$

$f_\lambda = 0$ unless the representation λ is self conjugate. If the representation λ is self conjugate , there exist an invariant bilinear form. Then $f_\lambda = 1$ if this bilinear form is symmetric , and $f_\lambda = -1$ if it is antisymmetric [13].

Now consider a sphere with $r' + n$ boundaries. If one turns r' boundaries into RP^2 , the connected sum of a sphere with n boundaries and $r' = 2g + 2s + r$ RP^2's is obtained. In this way the wave function is obtained to be :

$$\psi(g_1, ..., g_n) = \int dg'_1...dg'_{r'} \sum_\lambda d(\lambda)^{2-r'-n}\chi_\lambda(g_1'^2)...\chi_\lambda(g_r'^2)\chi_\lambda(g_1)...\chi_\lambda(g_n)e^{-\frac{\epsilon}{2}c_2(\lambda)A}$$

$$= \sum_{\lambda} f_{\lambda}^{r'} d(\lambda)^{2-r'-n} \chi_{\lambda}(g_1)...\chi_{\lambda}(g_n) e^{-\frac{\epsilon}{2}c_2(\lambda)A}. \tag{10}$$

Note that this surface is the most general nonorientable surface with boundaries , because it is well known that a surface with g handles , s Klein bottles , and r projective planes , is in fact nothing but a sphere with $r' = r + 2(g+s)$ projective planes (provided r and s are not both zero). Comparing (10) with (4) , shows this once again : (4) is a special case of (10) , with $n = 0$ and $r' = 2(g+s)$. To summarize , it is shown that

$$\psi(\Sigma_{g,s,r}, g_1, ..., g_n) = \sum_{\lambda} f_{\lambda}^{r+2s} d(\lambda)^{2-2g-2s-r-n} \chi_{\lambda}(g_1)...\chi_{\lambda}(g_n) e^{-\frac{\epsilon}{2}c_2(\lambda)A}, \tag{11}$$

and this relation holds for orientable , as well as nonorientable surfaces.

Now everything is ready to calculate the n-point functions of field strength on $\Sigma_{g,s,r}$. It was shown in [12] that the wave function on a disk , with boundary condition g_1 , in the presence of a source function $J(x)$ and in Scwhinger-Fock gauge is (this equation can also be found in the appendix of [11])

$$\psi_D[J] = \int \mathcal{D}\xi e^{-\frac{1}{2\epsilon}\int \xi^a \xi_a d\mu + \int \xi^a J_a d\mu} \delta(\text{Pexp} \oint_{\gamma} A, g_1)$$

$$= Z_1[J] \psi_{2,D}[J], \tag{12}$$

where

$$Z_1[J] = e^{\frac{\epsilon}{2}\int J^a J_a d\mu}, \tag{13}$$

and

$$\psi_{2,D}[J] = \sum_{\lambda} \chi_{\lambda}(g_1^{-1}) e^{-\frac{\epsilon}{2}c_2(\lambda)A(D)} \chi_{\lambda}(\mathcal{P}\text{exp}\epsilon \int dt (\int ds \sqrt{g} J(s,t))). \tag{14}$$

$\xi(x) = \xi^a(x) T^a$ is defined by $F_{\mu\nu} = \xi(x)\epsilon_{\mu\nu}$, T^a's are the generators of G and $d\mu = \sqrt{g(x)} d^2x$. The ordering in (14) is according to t. the disk is parametrized by coordinates s (the radial coordinate) and t (the angle coordinate). The functinal derivative of the above wave functionals produce the Green functions , where for $\psi_D[J]$ is :

$$< \xi^{a_1}(x_1)...\xi^{a_n}(x_n) >_{2,D} = \sum_{\lambda} \epsilon^n \chi_{\lambda}(g_1^{-1}) \chi_{\lambda}(T^{a_1}...T^{a_n}) e^{-\frac{\epsilon}{2}c_2(\lambda)A(D)} \tag{15}$$

for $\quad t(x_1) < ... < t(x_n)$.

Now to calculate the n-point function of ξ^a's on $\Sigma_{g,s,r}$ it is enough to glue the expectation value (15) to the wave function (11) with $n = 1$. The result is :

$$< \xi^{a_1}(x_1)...\xi^{a_n}(x_n) >_{2,\Sigma_{g,s,r}} = \frac{1}{Z_{\Sigma_{g,s,r}}} \int dg_1 < \xi^{a_1}(x_1)...\xi^{a_n}(x_n) >_{2,D} \psi(\Sigma_{g,s,r}, g_1)$$

$$= \frac{1}{Z_{\Sigma_{g,s,r}}} \sum_{\lambda} \epsilon^n f_{\lambda}^{r+2s} d(\lambda)^{1-2g-2s-r} \chi_{\lambda}(T^{a_1}...T^{a_n}) e^{-\frac{\epsilon}{2}c_2(\lambda)A}. \tag{16}$$

The Green function corresponding to Z_1 , which is the partition function of a free field theory , are [12] :

$$< \xi^{a_1}(x_1)...\xi^{a_{2n}}(x_{2n}) >_{1,\Sigma_{g,s,r}} = \sum_p G^{a_{i_1}a_{i_2}}(x_{i_1}, x_{i_2})...G^{a_{i_{2n-1}}a_{i_{2n}}}(x_{i_{2n-1}}, x_{i_{2n}}) \qquad (17)$$

where

$$G^{ab}(x,y) = \epsilon\delta^{ab}\delta(x-y), \qquad (18)$$

and the summation is over all distinct pairing of $2n$ indeces. The complete n-point functions are :

$$< \xi(x_1)...\xi(x_n) >= \sum_{m=0}^{n}\sum_c < \xi(x_1)...\xi(x_m) >_1 < \xi(x_{m+1})...\xi(x_n) >_2, \qquad (19)$$

where the inner summation is over all different ways of choosing m indeces from $2n$ indeces. It is clear that the correlators consist of a free (eq.(17)) and an almost x-independent part (eq.(16)). This is the same behaviour that was obsereved in the orientable case [12].

The last question that must be answered is that which part of the above Green functions are gauge invariant. In ref.[12] , by two methods , we showed that only the free part is gauge invariant. The first method was that if one constructs the gauge invariant wave function on disk , $\psi_D^{G.I.}[J]$, it can be shown that their corresponding Green functions are those quoted in (17). The same argument holds here and therefore in nonorientable case the gauge invariant n-point functions correspond to a free field theory. The second method is to calculate the Green functions by a different approach , that is by using the expectation value of Wilson loops , which is gauge invariant. So we must first calculate these expectation values. It is clear that it can be constructed as follows

$$< \chi_\mu(\text{Pexp}\oint_\gamma A) >= \frac{1}{Z_{g,s,r}}\int \psi(\Sigma_{g,s,r}, g_1)\chi_\mu(g_1)\psi_D(g_1^{-1})dg_1, \qquad (20)$$

where $\gamma = \partial D$ is a contractable loop. A simple calculation shows

$$< \chi_\mu(\text{Pexp}\oint_\gamma A) >= \frac{1}{Z_{g,s,r}}\sum_\lambda\sum_{\rho\in\lambda\otimes\mu} f_\lambda^{r+2s} d(\rho)d(\lambda)^{1-2g-2s-r} exp\{-\frac{\epsilon}{2}[c_2(\lambda)(A-A(D))+c_2(\rho)A(D)]\} \qquad (21)$$

In small $A(D)$ limit , the LHS of (21) becomes

$$\text{LHS} = d(\mu) + \frac{1}{2}\int \chi_\mu < \xi(x)\xi(y) > d\mu(x)d\mu(y) + ... \qquad (22)$$

Now if by symmetry consideration we use the following ansatz for the gauge invariant two-point function

$$< \xi^a(x)\xi^b(y) >^{G.I.} = M\delta^{ab}\delta(x-y), \qquad (23)$$

then eq.(22) reduces to

$$\text{LHS} = d(\mu) - \frac{M}{2}A(D)d(\mu)c_2(\mu). \qquad (24)$$

In the same limit , it can be shown that the RHS of eq.(21) is [12]

$$\text{RHS} = d(\mu) - \frac{\epsilon}{2} A(D) d(\mu) c_2(\mu), \tag{25}$$

which shows that $M = \epsilon$ and is consistent with eq.(18) .

At the end , we must note that if the gauge group was $U(1)$, we would produce nothing for nonorientable surfaces , for the simple reason that non of the representation of $U(1)$ are self conjugate except for the trivial representation.

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